The p-adic zeta function and a p-adic Euler constant

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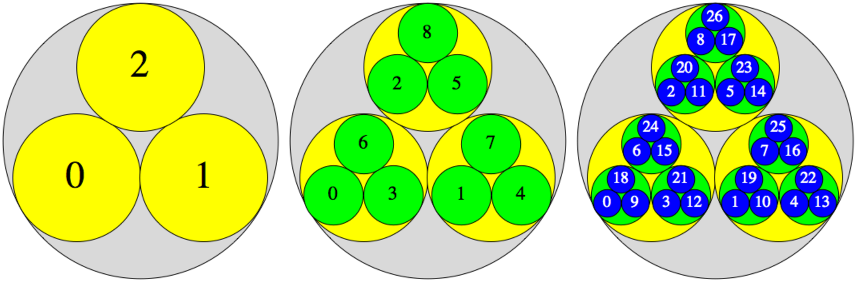
Topics of this Presentation

- Introduction to zeta functions and p-adic numbers
- Definitions of the p-adic zeta function ζ_p
- The Euler-Mascheroni constant γ_p
- Computations of γ_p

The p-adic numbers

Let $\mathbb{Z}_p = \varprojlim(\mathbb{Z}/p^n)$ and $\mathbb{Q}_p = Quot(\mathbb{Z}_p)$ the quotient field. $\mathbb{Q}_p$ is a complete non-Archimedean valued field with absolute value $|\ |_p$ such that $|p| = \frac{1}{p}$. The corresponding topology on $\mathbb{Q}_p$ is zero-dimensional (basis of clopen sets); the balls $x + p^n\mathbb{Z}_p$, where $x \in \mathbb{Q}_p$ and $n \in \mathbb{Z}$, are open and compact.



Topology of $\mathbb{Z}_3$

Zeta and L-functions

Define a p-adic analogue ζ_p of the complex Riemann zeta function $\zeta(s)$ and Dirichlet L-functions $L(s,\chi)$, where χ is a Dirichlet character:

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}\ , \qquad L(s,\chi) = \sum_{n\geq 1} \frac{\chi(s)}{n^s}\ , \qquad s > 1$$

Consider the Dirichlet characters $\chi = \omega^i : (\mathbb{Z}/p\mathbb{Z})^* \to \mu_{p-1}$ for $i = 0, \ldots, p-2$, where ω is the Teichmüller character.
We get $p-1$ twisted zeta functions:

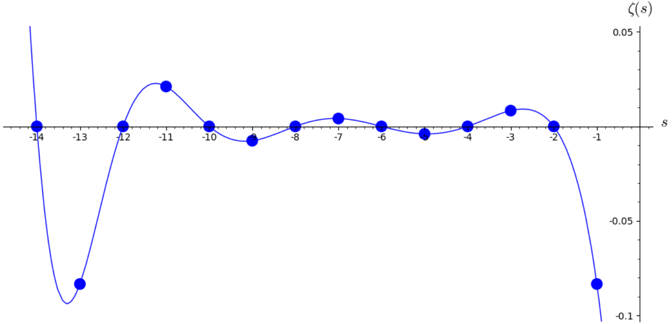
$$\zeta(s,\omega^i) = L(s,\omega^i)$$

Zeta function at negative integers

The primary way of defining a p-adic L-functions is via p-adic interpolation of special values of classical L-functions. It is well known that

$$\zeta(1-k) = -\frac{B_k}{k} \in \mathbb{Q}$$

for integers $k \geq 1$, where B_k are the Bernoulli numbers.



p-adic Interpolation

Can we find a continuous function $\zeta_p : \mathbb{Z}_p \to \mathbb{Q}_p$ which interpolates $\zeta(s)$ at $s = -1, -2, -3, \ldots$? Not quite, we have to:

- remove the Euler factor $(1 - p^{-s})^{-1}$, and
- define $p - 1$ branches $\zeta_{p,i}$.

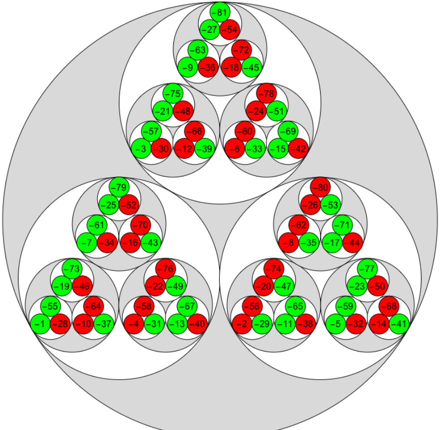
The branches $\zeta_{p,i}(s) = L_p(s, \omega^{1-i})$ are continuous (Kummer congruences) and interpolate $(1 - p^{-s})\zeta(s, \omega^{1-i-k})$ for $s = 1 - k$ and $k \geq 2$. For $k \equiv 1 - i \mod p - 1$, we have

$$\zeta_{p,i}(1 - k) = (1 - p^{k-1})\zeta(1 - k) = -(1 - p^{k-1})\frac{B_k}{k}.$$

For each residue class $i \in \mathbb{Z}/(p-1)\mathbb{Z}$, the negative integers $1 - k = -1, -2, -3, \ldots$ with $k \equiv 1 - i \mod p - 1$ are *dense* in $\mathbb{Z}_p$.

Example

Let $p = 3$. The sets of negative integers $1 - k = -1, -2, -3, \ldots$ with $1 - k \equiv 0 \mod 2$ (red) and $1 - k \equiv 1 \mod 2$ (green) are *dense* in $\mathbb{Z}_3$.



p-adic Integration and Bernoulli numbers

Now show that $\zeta_{p,i}$ is analytic. It follows from elementary formulas on sums of powers that

$$\int_{\mathbb{Z}_p} x^k dx = B_k.$$

Here we use the Volkenborn integral, which is based on the Haar distribution on $\mathbb{Z}_p$. Furthermore, we have

$$\int_{\mathbb{Z}_p^*} x^k dx = (1 - p^{k-1})B_k = -k\zeta_{p,i}(1-k).$$

for integers $k \geq 2$ and $k \equiv 1 - i \mod p - 1$.

Kubota-Leopoldt L-function

The domain of ζ_p can be extended to a subset of $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$, the completion of the algebraic closure of $\mathbb{Q}_p$. The Tate field $\mathbb{C}_p$ is the p-adic analogue of the complex numbers $\mathbb{C}$.

Basically, we use the above formula

$$\int_{\mathbb{Z}_p^*} x^k dx = (1 - p^{k-1})B_k = -k\,\zeta_{p,i}(1-k).$$

and replace $k \in \mathbb{Z}$ by $1 - s \in \mathbb{C}_p$. However, the function x^{1-s} is not continuous in s unless $x \equiv 1 \mod p$. So we set $\langle x \rangle = \frac{x}{\omega(x)}$ and define

$$\zeta_{p,i}(s) = \zeta_p(s, \omega^{1-i}) = \frac{1}{s-1} \int_{\mathbb{Z}_p^*} \omega(x)^{1-i} \langle x \rangle^{1-s} dx$$

for $s \in \mathbb{C}_p$ such that $|s| < p^{\frac{p-2}{p-1}}$.

p-adic Measures

The Kubuta-Leopoldt L-function uses the Haar *distribution*. Now define p-adic zeta functions using a *measure* on $\mathbb{Z}_p$. Firstly, one considers Bernoulli distributions. Secondly, the Bernoulli distributions are *regularized* and turned into measures.

Theorem

Let $p \neq 2$ be a prime, $n \in \mathbb{N}$ and $a \in \{0, 1, \ldots, p^n - 1\}$. Then

$$\mu(a + p^n\mathbb{Z}_p) = \begin{cases} \frac{1}{2} & \text{if } a \text{ is even} \\ -\frac{1}{2} & \text{if } a \text{ is odd} \end{cases}$$

defines a measure μ on $\mathbb{Z}_p$. This measure is the regularization of the first Bernoulli distribution for $c = 2$.

p-adic measures and Bernoulli numbers

Theorem

Let $k \in \mathbb{N}$ and μ the above measure on $\mathbb{Z}_p$. Then

$$\int_{\mathbb{Z}_p} x^{k-1} d\mu = \frac{B_k}{k}(1-2^k).$$

Restricting the integration to $\mathbb{Z}_p^*$ yields

$$\int_{\mathbb{Z}_p^*} x^{k-1} d\mu = \frac{B_k}{k}(1-2^k)(1-p^{k-1}).$$

$\zeta_{p,i}$ can be constructed using μ:

$$\zeta_{p,i}(s) = \zeta_p(s, \omega^{1-i}) = -\frac{1}{1-\omega(2)^{1-i}\langle 2\rangle^{1-s}} \int_{\mathbb{Z}_p^*} \omega(x)^{-i} \langle x\rangle^{-s} d\mu$$

Iwasawa functions

$\zeta_{p,i}(s)$ is a Mellin transform of μ and analytic for $s \neq 1$. Furthermore, the integral representation shows that $\zeta_{p,i}(s)$ is a power series in $(1+p)^s$ with coefficients in $\mathbb{Z}_p$ if $i \neq 1$. For $i = 1$, i.e., if ω^{1-i} is the trivial character, $(s-1)\zeta_{p,1}(s)$ is a power series. Such functions are called Iwasawa functions:

$$f((1+p)^s - 1) \text{ where } f \in \mathbb{Z}_p[[1 + p\mathbb{Z}_p]] \cong \mathbb{Z}_p[[T]]$$

There is an algebraic construction of f using ideal class groups elements of the cyclotomic fields $\mathbb{Q}(\zeta_{p^n})$ and the *Main Conjecture* (a theorem in this case) relates the two types of zeta functions.

Classical Euler constant

The Euler constant $\gamma \approx 0.5772$ has many appearances including:

- γ is the asymptotic difference between the harmonic series and the logarithm:

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n)\right)$$

- γ is the constant coefficient of the Laurent expansion of $\zeta(s)$ about $s = 1$:

$$\zeta(s) = \frac{1}{s-1} + \gamma + \ldots$$

- γ is related to the derivative of the gamma function Γ at $s = 1$:

$$\gamma = -\Gamma'(1)$$

p-adic Euler constant

The p-adic Euler constant was first defined by J. Diamond and the relation to ζ_p is due to N. Koblitz.

Definition

The p-adic Euler constant γ_p is the constant coefficient of the 1-branch $\zeta_{p,1}$ of the p-adic zeta function about $s = 1$:

$$\zeta_{p,1}(s) = \frac{1 - \frac{1}{p}}{s - 1} + \gamma_p + \dots$$

Computation using the Kubota-Leopoldt L-function

Recall that $\zeta_{p,1}(s) = \zeta_p(s, \omega^0) = \frac{1}{s-1} \int_{\mathbb{Z}_p^*} \langle x \rangle^{1-s} dx$. We want to compute the constant term; the residual is $1 - \frac{1}{p}$.

$$\gamma_p = \lim_{s \to 1} \frac{1}{s-1} \left(\int_{\mathbb{Z}_p^*} \langle x \rangle^{1-s} dx - \left(1 - \frac{1}{p}\right)\right)$$
$$= \lim_{n \to \infty} -\frac{1}{p^n} \left(\frac{1}{p^n} \sum_{x=0,\ p \nmid x}^{p^n} \langle x \rangle^{p^n} - \left(1 - \frac{1}{p}\right)\right)$$

Since $\langle x \rangle^{p^n} = \omega^{-1} x^{p^n}$, the sum can be computed using generalized Bernoulli numbers. This gives

$$\gamma_p = \lim_{n \to \infty} -\frac{1}{p^n} \left(B_{p^n, \omega^{-1}} - \left(1 - \frac{1}{p}\right)\right).$$

γ_p and the $p-$adic logarithm

We use the expansion of $\langle x\rangle^{1-s}$:

$$\langle x\rangle^{1-s} = \exp_p((1-s)\log_p\langle x\rangle) = \sum_{n=0}^{\infty}(\log_p\langle x\rangle)^n \frac{(1-s)^n}{n!}$$

The power series can be integrated termwise and

$$\zeta_{p,1}(s) = \frac{1}{s-1}\int_{\mathbb{Z}_p^*}\langle x\rangle^{1-s}dx$$
$$= \frac{1}{s-1}\sum_{n=0}^{\infty}\left(\int_{\mathbb{Z}_p^*}(\log_p\langle x\rangle)^n dx\right)\frac{(1-s)^n}{n!}.$$

This yields Diamond's original definition of γ_p:

$$\gamma_p = \int_{\mathbb{Z}_p^*} -\log_p\langle x\rangle dx = \lim_{n\to\infty} -\frac{1}{p^n}\sum_{x=0,\ p\nmid x}^{p^n}\log_p\langle x\rangle$$

γ_p and the $p-$adic gamma function

We can rewrite the sum of the logarithms as

$$\sum_{x=0,\ p\nmid x}^{p^n} \log_p(x) = \log_p\left(\prod_{x=0,\ p\nmid x}^{p^n} x\right) = \log_p(\Gamma_p(p^n)).$$

This gives:

$$\gamma_p = \lim_{n\to\infty} -\frac{1}{p^n}\log_p(\Gamma_p(p^n)) = -(\log_p \Gamma_p)'(0) = \Gamma_p'(1).$$

Γ_p interpolates the factorial with the factors dividing p removed:

$$\gamma_p = \lim_{n\to\infty} \frac{1}{p^n}\left(\Gamma_p(p^n+1) - \Gamma_p(1)\right) = \lim_{n\to\infty} \frac{1}{p^n}\left(\frac{p^n!}{p^{n-1}!\,p^{p^{n-1}}} + 1\right)$$

Computation using $p-$adic measures

We have the formula

$$\zeta_{p,1}(s) = -\frac{1}{1-\langle 2\rangle^{1-s}} \int_{\mathbb{Z}_p^*} \omega(x)^{-1} \langle x\rangle^{-s} d\mu$$

Now we use the definition of μ and obtain:

$$\zeta_{p,1}(s) = -\frac{1}{1-\langle 2\rangle^{1-s}} \lim_{n\to\infty} \sum_{x=0,\, p\nmid x}^{p^n} \omega(x)^{-1} \langle x\rangle^{-s} \frac{(-1)^x}{2}$$

Dirichlet series expansion

We have a Dirichlet series expansion of the p-adic zeta function:

$$\zeta_{p,i}(s) = \frac{-1}{1-\omega(2)^{1-i}\langle 2\rangle^{1-s}} \lim_{n\to\infty} \sum_{\substack{m=0 \\ p\nmid m}}^{p^n} \omega(m)^{-i}\langle m\rangle^{-s} \frac{(-1)^m}{2}$$

Set $s = 1 - p^n$. Then

$$\gamma_p = \lim_{n\to\infty} \frac{-1}{1-\langle 2\rangle^{p^n}} \left(\sum_{\substack{m=0 \\ p\nmid m}}^{p^n} \omega(m)^{-1}\langle m\rangle^{p^n-1} \frac{(-1)^m}{2} \right) + \frac{1}{p^n}\left(1-\frac{1}{p}\right).$$

For $k \equiv i \mod p-1$ and $s = k \in \mathbb{Z}$, we get D. Delbourgo's result:

$$\zeta_{p,i}(k) = \frac{-1}{1-2^{1-k}} \cdot \lim_{n\to\infty} \sum_{\substack{m=1 \\ p\nmid m}}^{p^n} \frac{(-1)^m}{2} m^{-k}$$

$$= (1-p^{-k})\zeta(k) \text{ for } k < 0.$$

Computations of γ_p

Using any of the above formulas for γ_p and SageMath software, we obtain (like D. Delbourgo):

$$\gamma_3 = 2\cdot 3 + 2\cdot 3^2 + 3^3 + 2\cdot 3^4 + 3^5 + 2\cdot 3^6 + 2\cdot 3^7 + 2\cdot 3^8 + O(3^{10})$$
$$\gamma_5 = 5 + 3\cdot 5^3 + 2\cdot 5^5 + 3\cdot 5^6 + 4\cdot 5^7 + 5^8 + 2\cdot 5^9 + O(5^{10})$$
$$\gamma_7 = 5 + 2\cdot 7 + 4\cdot 7^2 + 6\cdot 7^3 + 2\cdot 7^4 + 6\cdot 7^6 + 2\cdot 7^7 + 7^9 + O(7^{10})$$
$$\gamma_{11} = 1 + 10\cdot 11 + 2\cdot 11^2 + 11^3 + 5\cdot 11^4 + 5\cdot 11^5 + 4\cdot 11^6 + O(11^7)$$
$$\gamma_{13} = 4\cdot 13 + 7\cdot 13^3 + 8\cdot 13^4 + 7\cdot 13^5 + 6\cdot 13^6 + 4\cdot 13^7 + O(13^8)$$